Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

An analytic derivation of clustering coefficients for weighted networks

Yichao Zhang[1], Zhongzhi Zhang[2,3], Jihong Guan[1] and Shuigeng Zhou[2,3]

[1] Department of Computer Science and Technology, Tongji University, 4800 Cao'an Road, 201804, Shanghai, People's Republic of China
[2] School of Computer Science, Fudan University, 200433, Shanghai, People's Republic of China
[3] Shanghai Key Laboratory of Intelligent Information Processing, Fudan University, 200433, Shanghai, People's Republic of China
E-mail: 2008zhangyichao@tongji.edu.cn, zhangzz@fudan.edu.cn, jhguan@mail.tongji.edu.cn and sgzhou@fudan.edu.cn

Received 3 November 2009
Accepted 9 February 2010
Published 16 March 2010

Online at stacks.iop.org/JSTAT/2010/P03013
doi:10.1088/1742-5468/2010/03/P03013

Abstract. Clustering coefficients are among the most important parameters characterizing the topology of complex networks and have a significant influence on various dynamical processes occurring on networks. On the other hand, a plethora of real-life networks with diverse links can be described better in terms of weighted networks than in terms of binary networks, where all links are homogeneous. However, analytical research on clustering coefficients in weighted networks is still lacking. In this paper, we apply an extended mean-field approach to investigate clustering coefficients for the typical weighted networks proposed by Barrat, Barthélemy and Vespignani (BBV networks) (2004 *Phys. Rev. Lett.* **92** 228701). We provide an analytical solution to the model, showing how the local clustering of a node in the BBV networks depends on its degree and strength. Our analysis is in good agreement with the results of numerical simulations.

Keywords: disordered systems (theory), exact results

Contents

1. Introduction

In the past decade, studies on relatively simple binary (Boolean) networks where edges (links) are either present or absent [1]–[4] took a successful step in the direction of understanding the structure and function of complex networks. However, the connections in many real networks are not homogeneous, which naturally calls for a more sophisticated form, weighted networks [5]–[8], for better modeling these systems, where each connection is assigned a *weight* value to denote a physical property of interest. For a social network, the link weight indicates the intimacy between two individuals [9, 10]; for the Internet, the link weight describes the bandwidth of an optical cable between two routers [11]; for the worldwide aviation network, the link weights reflect the annual volume of passengers traveling between two airports [5]; and for the *E. coli* metabolic network, the link weight encodes the optimal metabolic fluxes between two metabolites [12]. Consequently, weighted networks encode more information and they are a more realistic representation of real systems where individual links (and nodes) are vastly different.

Previous studies on weighted networks have indicated the importance of the link weights of networks and led to the formulation of a long list of models better mimicking real systems [5]–[7], [13]–[26]. The first evolving weighted network was proposed by Yook *et al* [13], where the topology and weight are driven simultaneously by the preferential attachment rule [27]. However, to some extent, the evolutionary mechanism of weighting for the model did not match the real processes well. Instead, Barrat, Barthélemy, and Vespignani presented a growing model (BBV) [5]–[7] for weighted networks, where the topology and weight coevolve interdependently. Motivated by BBV's remarkable work, a variety of models and mechanisms for weighted networks have been proposed. For example in a weight-driven model [14], each link carries a positive weight which is drawn from a certain distribution; in a fitness model [15], the weight and topology evolve independently; in traffic-driven evolution models [16]–[19], the evolution of weight and the topology mutually interact. Besides this, accelerating growth [20], deterministic models [21]–[23], weight-dependent deactivation [24], and spatial constraints [25, 26] provide some interesting insight as well.

In the light of the influence on the studies of weighted networks mentioned above, the BBV model has been acknowledged to be a pioneer and classic work in physics

J. Stat. Mech. (2010) P03013

communities. The main contribution of the work can be ascribed to two aspects: a new strength-driven preferential attachment mechanism; and local weight and strength rearrangements for a new link. The results from this model represent many behaviors observed empirically in real-world networks, e.g., link weight–degree correlation, node strength–degree correlation, and scale-free topology. Thus, it is considered as a general starting point for more realistic models as a representation of specific networks.

As is known to us all, a primary purpose of the current studies of complex networks is to understand how their dynamical behaviors are influenced by underlying geometrical and topological properties [4, 28]. Among many fundamental structural characteristics [29], the clustering coefficient, also known as the transitivity, plays an important role in the theory of complex networks and has attracted considerable attention. In the sociological literature, for example, it is referred to as the 'network density'. For a node i with k_i links, its clustering coefficient C_i denotes the fraction of these allowable links that actually exist. The average clustering coefficient for the whole system is defined as the average of C_i for each node [30]. This quantity is very important because it is one of the two key criteria for judging whether a network or a complex system exhibits the small-world property [30]. Also, it has a profound influence on a variety of crucial fields, such as those of epidemic processes [31]–[33], network resilience [34, 35] and community structure detection [36, 37], to name but a few. Very recently, on the basis of certain distributions of edges and triangles, Newman were the first to provide an exact solution for the parameter [38] in an interesting random graph. This work contributed an invaluable tool for investigating the coefficients for some random graph models [39]–[41]. Unfortunately, the recipe cannot be adopted directly for solving problems in networks where the connections are not randomly created and the distributions of edges and triangles cannot be acquired easily, for example, BA networks [27] and small-world networks [30]. So far, most previous related studies on these networks have been confined to numerical methods, which is, however, prohibitively difficult for large networks because of the limits of time and computer memory. Hence, it is necessary to present an analytical solution for the clustering coefficient, toward providing some useful insight into topological structures of weighted networks.

Despite its importance, to the best of our knowledge, the rigorous computation for the clustering coefficient of the BBV model has not been addressed yet, contrasting sharply with the successes as regards other network properties such as the degree, weight and strength distribution [7]. To fill this gap, in this present paper we investigate this interesting quantity analytically. We derive an implicit formula for the clustering coefficient characterizing the BBV model. The analytic method adopted in this work is an extended mean-field approach. The precise result obtained shows that the average local clustering coefficient of the BBV network decreases as a power law in the network order, the total number of nodes in the networks. To some extent, our research opens a way to theoretically studying the clustering coefficient of neutral or nonassortative mixing [42] weighted networks. At the same time, our analytical solution may give insight different from that afforded by the approximate solution of a numerical simulation.

2. Introduction to the BBV model

The BBV model starts from an initial seed of N_0 vertices fully connected by links with assigned weight w_0. A new vertex n is added at each time step. This new site is connected

J. Stat. Mech. (2010) P03013

to m previously existing vertices (i.e., each new vertex will have initially exactly m links, all with equal weight w_0), choosing preferentially sites with large strength. In other words, an old node i is chosen according to the following probability:

$$\Pi_{n\rightarrow i} = \frac{s_i}{\Sigma_v s_v} \tag{1}$$

where s_i is the strength of node i, defined as $s_i = \sum_j w_{ij}$. The weight w_{in} of the new link L_{in} connecting nodes n and i is initially set to a given value w_0. The creation of this edge will introduce a local variations of the edge weight $w_{ij} \rightarrow w_{ij} + \Delta_{w_{ij}}$, in which node j is one of the neighbors of node i and its node strength $s_j \rightarrow s_j + \Delta_{w_{ij}}$, where this perturbation is proportionally distributed among the links according to their weights:

$$\Delta_{w_{ij}} = \delta_i \frac{w_{ij}}{s_i}. \tag{2}$$

Simultaneously, this rule yields a total strength increase for node i of $\delta_i + w_0$, implying that $s_i \rightarrow s_i + \delta_i + w_0$. In this paper, we focus on the case of homogeneous coupling with $\delta_i = \delta = \text{const}$ and equivalently set $w_0 = 1$. After the weights and strengths have been updated, the growth process is iterated by introducing a new vertex, until the desired size of the network is reached.

In the BBV networks, the dynamical process for s_i and k_i can thus be described through the following evolution:

$$\frac{\mathrm{d}s_i}{\mathrm{d}t} = m\frac{s_i(t)}{\Sigma_v s_v}(1+\delta) + \Sigma_{j\in\Omega(i)} m \frac{s_i(t)}{\Sigma_v s_v}\delta\frac{w_{ij}(t)}{s_j(t)}, \tag{3}$$

$$\frac{\mathrm{d}k_i}{\mathrm{d}t} = m\frac{s_i(t)}{\Sigma_v s_v}. \tag{4}$$

Note that $\sum_{j\in\Omega(i)}$ runs over the nearest neighbors of the node i. Solving these two equations with initial conditions $k_i(t_i) = s_i(t_i) = m$, one has

$$s_i = m\left(\frac{t}{t_i}\right)^{(2\delta+1)/(2\delta+2)} \tag{5}$$

(reference [7]) and

$$k_i = \frac{s_i + 2m\delta}{2\delta+1}. \tag{6}$$

After introducing the BBV model, we will investigate analytically the clustering coefficient for all the nodes in the BBV networks. As will be shown, like for the Barabási–Albert (BA) networks [27], the average clustering coefficient of the BBV networks rapidly decreases with increasing network size. In previous work [7], researchers have presented numerical calculations reporting this result, while we will offer an exact solution to explain it in what follows.

3. An analytical expression for clustering coefficients of the model

Before calculating the clustering coefficient, we first introduce a basic concept—'what neutral weighted networks are'. In networks, the degree of a node is defined as the number of links that the node has. A relevant topological property is the degree–degree correlation, or so-called assortative mixing. It is defined as a preference for high-degree vertices to attach to other high-degree vertices and vice versa [42, 43]. In contrast, disassortative mixing is defined as a preference for high-degree vertices to preferentially connect with low-degree nodes and vice versa. However, a network having neutral degree correlation means that it is neither assortative nor disassortative. A random network, where links are randomly connected between nodes, is an example of a neutral network.

We introduce, along with the degree correlations, the weighted degree correlations [5], defined as

$$k^{\mathrm{w}}_{\mathrm{nn},i} = \frac{1}{s_i}\sum_{j=1}^{N} a_{ij} w_{ij} k_j, \tag{7}$$

which measure the effective affinity for connecting with high-degree or low-degree neighbors according to the magnitude of the actual interactions. Also, the behavior of the function $k^{\mathrm{w}}_{\mathrm{nn}}(k)$ marks the weighted assortative or disassortative properties. When $k^{\mathrm{w}}_{\mathrm{nn}}(k)$ increases with k, it means that nodes have a tendency to connect to nodes with a similar or larger degree. In this case the network is defined as assortative. In contrast, if $k^{\mathrm{w}}_{\mathrm{nn}}(k)$ is decreasing with k, which implies that nodes having large degree are likely to have near neighbors with small degree, then the network is said to be disassortative. Naturally, if correlations are absent, this can be ascribed to weighted neutral networks.

Of particular interest is the BBV model mentioned in section 2 which can give rise to neutral networks for small δ [7]. However, for increasing δ, the disassortative character and a power law behavior of the $k_{\mathrm{nn}}(k)$ emerge. Fortunately, the weighted assortativity $k^{\mathrm{w}}_{\mathrm{nn}}(k)$ is a constant, that is, they belong to weighted neutral networks. In this case, we redefine the linking probability of nodes i and j,

$$p_{ij} = \frac{s_j s_i}{\sum_v s_v} = \frac{s_j s_i}{2m(1+\delta)t w_{ij}}, \tag{8}$$

in order to extend the mean-field method to this particular network for large δ.

In a BBV network, consider a certain node i at time step t. Note that we only consider the case $m > 1$, in that the BBV network is close to tree-like networks when $m = 1$. As is known to us all, for a tree network, the clustering coefficient in the network is equal to zero. By definition [30], the clustering coefficient $C_i = 2E_i/(k_i(k_i - 1))$ of node i depends on two variables, where E_i is the number of links between the k_i neighbors of i.

Since in the BBV model only new nodes may create links, the coefficient C_i changes only when its degree k_i changes, i.e., when new nodes create connections to i and $x \in \langle 0, m-1\rangle$ of its nearest neighbors. The appropriate equation for changes of C_i is then

$$\frac{\mathrm{d}C_i}{\mathrm{d}t} = \sum_{x=0}^{m-1} \tilde{p}_{ix} \Delta_{C_{ix}}, \tag{9}$$

J. Stat. Mech. (2010) P03013

where $\Delta_{C_{ix}}$ denotes the change of the clustering coefficient when a new node connects to the node i and to x of its first neighbors as well, whereas $\tilde{p}_{ix}$ describes the probability of this event.

Notice that, in the equation (9), $\Delta_{C_{ix}}$ represents the difference between clustering coefficients of the same node i calculated after and before a new node attachment:

$$\Delta_{C_{ix}} = \frac{2(E_i + x)}{k_i(k_i+1)} - \frac{2E_i}{k_i(k_i-1)} = \frac{2x}{k_i(k_i+1)} - \frac{2C_i}{k_i+1}, \tag{10}$$

where $\Delta_{C_{ix}} > 0$. Also, equation (10) provides the restricted region of our method. The probability $\tilde{p}_{ix}$ is a product of two factors:

$$\tilde{p}_{ix} = \frac{s_i}{\sum_v s_v} \binom{m-1}{x} P^x (1-P)^{m-1-x}. \tag{11}$$

The first factor is the probability for a new link to end up in i, which is given by equation (1). The second one is the probability that x among the rest of the $(m-1)$ new links connect to the nearest neighbors of i. It is equivalent to the probability of $(m-1)$ Bernoulli trials with the probability P for x successes. So,

$$P = \frac{\sum_{j\in\Omega(i)} s_j}{\sum_v s_v} = \frac{\sum_{j\in\Omega(i)} s_j}{2m(1+\delta)t}. \tag{12}$$

Replacing the sum $\sum_{j\in\Omega(i)}$ by an integral, one obtains

$$\sum_{j\in\Omega(i)} s_j \sim \int_1^t s_j p_{ij}\,\mathrm{d}t_j \tag{13}$$

where

$$p_{ij} = \frac{s_i s_j}{\sum_v s_v w_{ij}} = \frac{s_i s_j}{2m(1+\delta)t w_{ij}}, \tag{14}$$

in which w_{ij} is the weight of the link L_{ij}. In the BBV networks,

$$w_{ij}(t) = \left(\frac{t}{t_{ij}}\right)^{\delta/(\delta+1)}, \tag{15}$$

where $t_{ij} = \max(i, j)$. Hence, using equation (13), one can find the probability

$$P = \frac{mt^{(4\delta-1)/(2\delta+2)}}{4(1+\delta)\delta} \left(\mathrm{i}^{-(4\delta+1)/(2\delta+2)} - \frac{\mathrm{i}^{-(6\delta+1)/(2\delta+2)}}{2} - \frac{t^{-(2\delta+1)/(2\delta+2)}\mathrm{i}^{-(2\delta+1)/(2\delta+2)}}{2} \right). \tag{16}$$

Now, inserting equations (5), (6), (10) and (11) into equation (9) one can obtain

$$\frac{\mathrm{d}C_i}{\mathrm{d}t} + \frac{m(2\delta+1)t^{-1/(2\delta+2)}C_i}{(1+\delta)[mt^{(2\delta+1)/(2\delta+2)} + (2m\delta+2\delta+1)t_i^{(2\delta+1)/(2\delta+2)}]} = \frac{t^{-3/(2\delta+2)}}{2\delta(\delta+1)^2} \frac{m(m-1)(2\delta+1)}{mt^{(2\delta+1)/(2\delta+2)} + (2m\delta+2\delta+1)t_i^{(2\delta+1)/(2\delta+2)}}. \tag{17}$$

In the derivation process, we simplify

$$\frac{mt^{(2\delta+1)/(2\delta+2)} + (2m\delta+2\delta+1)\mathrm{i}^{(2\delta+1)/(2\delta+2)}}{mt^{(2\delta+1)/(2\delta+2)} + 2m\delta\mathrm{i}^{(2\delta+1)/(2\delta+2)}} \approx 1 \tag{18}$$

J. Stat. Mech. (2010) P03013

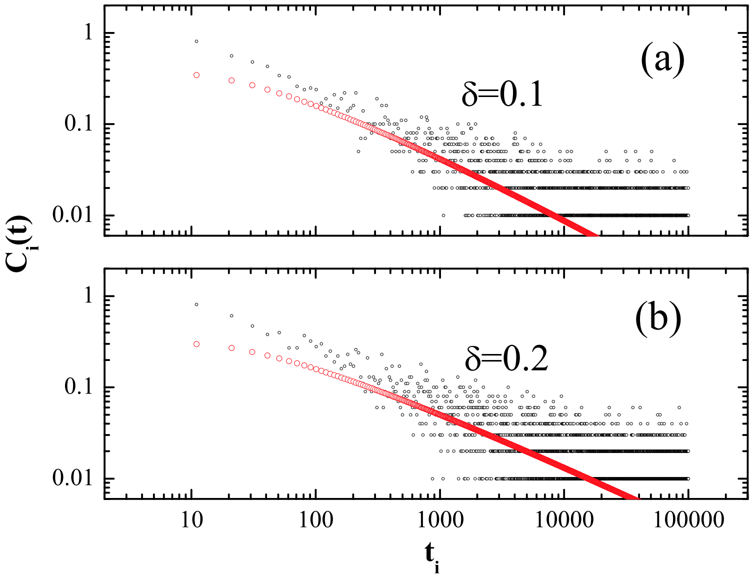

Figure 1. The initial value of the local clustering coefficient $C_i(t_i)$ for (a) $\delta = 0.1$ and (b) $\delta = 0.2$ (averaged over ten BBV networks). Red circles show the analytical solutions given by equation (20).

to make the analytical integral easily executable. Solving the equation for C_i one gets

$$C_i(t) = \left[mt^{(2\delta+1)/(2\delta+2)} + (2m\delta + 2\delta + 1)t_i^{(2\delta+1)/(2\delta+2)}\right]^{-2} \left[B_i + \frac{m^2}{4(1+\delta)\delta^2} \times \left(\frac{2t^{3\delta/(\delta+1)}t_i^{-(\delta/(\delta+1))} - t^{3\delta/(\delta+1)}t_i^{-(2\delta/(\delta+1))} - 6t^{\delta/(\delta+1)}}{6}\right)\right], \tag{19}$$

where B_i is an integration constant for the node i and determined by the initial condition $C_i(t_i)$ that describes the clustering coefficient of the node i exactly at the moment of its attachment t_i. Noting that $\Delta_{C_{ix}} > 0$ in equation (10), we have $1/k_i > C_i$. Thus, in this article, it is worth noticing that δ has to be confined to the region $[0, 1/4)$. The restriction is given by our approximation in equation (18). We believe that this drawback will be addressed in future work.

To clarify the calculation of $C_i(t_i)$, we show another form of the definition of C_i [28, 30]:

$$C_i = \frac{\text{number of triangles connected to vertex } i}{\text{number of triples centered on vertex } i}.$$

It can be rewritten as

$$C_i(t_i) = \frac{\sum_j \sum_v p_{ij}p_{iv}p_{jv}}{2\begin{pmatrix} m \\ 2 \end{pmatrix}} \sim \frac{m^2(1 - t_i^{-\delta/(\delta+1)})(1 - t_i^{-2\delta/(\delta+1)})t_i^{(\delta-1)/(\delta+1)}}{16(m-1)(\delta+1)\delta^2}. \tag{20}$$

Figure 1 shows the prediction from equation (20) in comparison with numerical results. For small values of t_i the numerical data differ from the theory significantly. This incongruity originates from the formula for the probability of a connection p_{ij} in

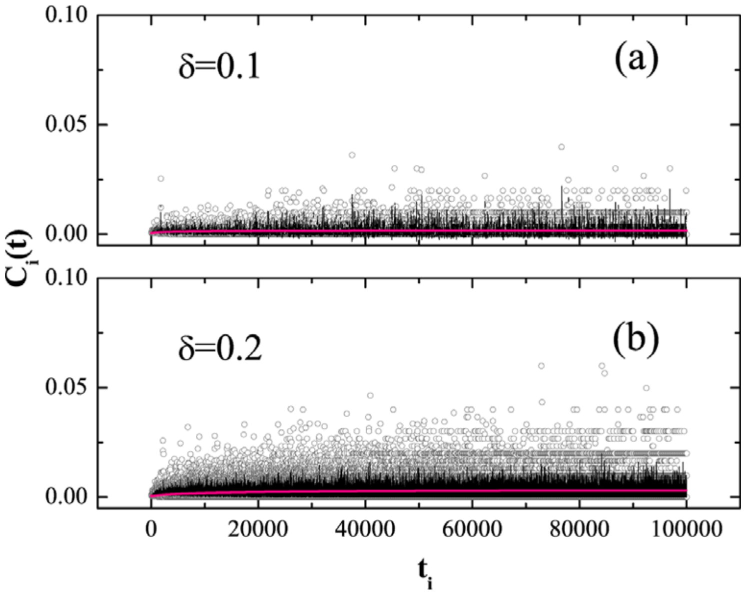

Figure 2. The local clustering coefficient $C_i(t)$ as a function t_i for (a) $\delta = 0.1$ and (b) $\delta = 0.2$. Results are obtained by numerical simulations on the BBV networks (averaged over ten networks). Red lines show the analytical solutions given by equation (19), and black curves show the smoothed form of the simulation data (gray circles).

equation (8) used three times in equation (20), which holds only in the asymptotic region $t_i \longrightarrow \infty$.

Taking into account the initial condition $C_i(t_i)$ from equation (20), one can obtain B_i for a given node i:

$$B_i = \frac{m^2(2m\delta + 2\delta + m + 1)^2(1 - t_i^{-\delta/(\delta+1)})(1 - t_i^{-2\delta/(\delta+1)})t_i^{3\delta/(\delta+1)}}{16(m-1)(\delta+1)\delta^2} - \frac{m^2}{4(1+\delta)\delta^2}\left(\frac{t_i^{2\delta/(\delta+1)}}{3} - \frac{7t_i^{\delta/(\delta+1)}}{6}\right). \tag{21}$$

Inserting B_i to equation (19), one can easily obtain $C_i(t)$. Notice that both numerical results and analytical solutions of $C_i(t)$ are shown in figure 2.

To obtain the average clustering coefficient C of the whole network, the expression of equation (21) has to be averaged over all nodes within a network. Admittedly, it is difficult to find an analytic result for C uniformly but we can show numerical solutions of BBV networks using the definition

$$C = \frac{\sum_{i=1}^{t} C_i(t)}{t}. \tag{22}$$

In figure 3, one can easily find that both the numerical and theoretical results exhibit a power law decay as the size of the networks grows. Hence, when t tends to infinity, C gets close to zero. As shown in figure 3, the power law exponent depends on the tunable parameter δ. For large δ, the decreasing velocity is relatively slow.

J. Stat. Mech. (2010) P03013

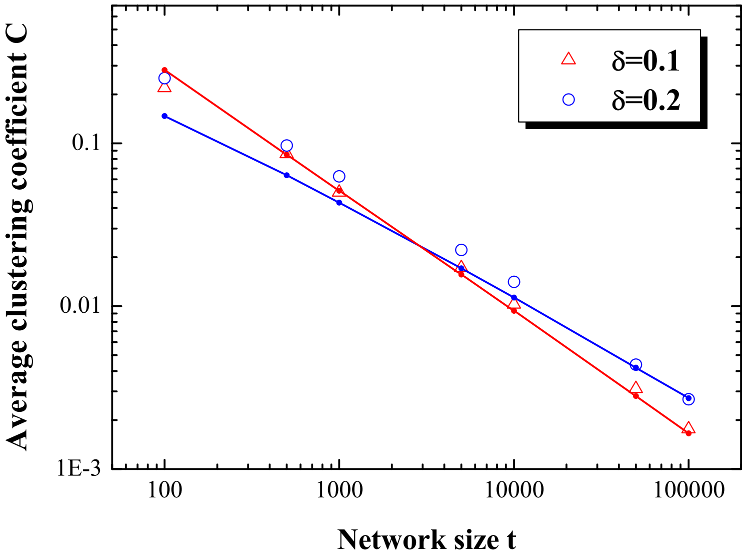

Figure 3. The clustering coefficient C of a whole BBV network as a function of the network size t for different values of δ. Results are obtained by numerical simulations on BBV networks (averaged over ten networks). The solid lines are numerical solutions given by equation (22).

Generally, the cluster coefficient plays a crucial role in the topology and dynamical processes on networks. Previous studies showed that for the BA networks [27], both local cluster coefficients $C_i(t)$ and their mean can be obtained using mean-field theory as the networks are uncorrelated [44]. For correlated networks, however, the mean-field method cannot be adopted simply. Hence, for most correlated networks, to present the cluster coefficient in an analytical way remains challenging. Admittedly, using an approach based on generating functions in the seminal work [38], Newman derived rigorous results for the parameter for a random graph. However, the key point of calculating the clustering coefficient lies in acquiring the distributions of these corresponding quantities first. Except for random graphs, for most normal networks (weighted or unweighted) reported, these distributions are uncertain and have to be calculated through numerical methods. Hence, comparatively few analytical investigations have been recorded in the literature relating to the parameter, both for random and deterministic weighted networks, as yet. For the BBV networks, we introduce an extended mean-field method for studying the property, as these networks are fortunately weight uncorrelated. Although specialized to the BBV networks here, this method may open up a new way to analytically investigating dynamics on the networks and their weight uncorrelated counterparts.

4. Conclusion

To sum up, the clustering coefficient is an important topological feature of complex networks. It has a profound impact in a variety of crucial fields, such as those of epidemic processes, network resilience and community detection, and so on. In this paper, we have derived analytically the solution for the clustering coefficient of the BBV model which has been attracting much research interest. We found that the average clustering coefficient decays as a power law function of the network order. Our analytical technique could guide and shed light on related studies for weighted networks by providing a paradigm for calculating the clustering coefficient.

Acknowledgments

This research was supported by the National Basic Research Program of China under grant No. 2007CB310806, the National Natural Science Foundation of China under Grants Nos 60704044, 60873040 and 60873070, Shanghai Leading Academic Discipline Project No. B114, and the Program for New Century Excellent Talents in University of China (NCET-06-0376).

References

[1] Albert R and Barabási A-L, 2002 *Rev. Mod. Phys.* **74** 47
[2] Dorogvtsev S N and Mendes J F F, 2002 *Adv. Phys.* **51** 1079
[3] Newman M E J, 2003 *SIAM Rev.* **45** 167
[4] Boccaletti S, Latora V, Moreno Y, Chavez M and Hwanga D-U, 2006 *Phys. Rep.* **424** 175
[5] Barrat A, Barthélemy M, Pastor-Satorras R and Vespignani A, 2004 *Proc. Nat. Acad. Sci.* **101** 3747
[6] Barrat A, Barthélemy M and Vespignani A, 2004 *Phys. Rev. Lett.* **92** 228701
[7] Barrat A, Barthélemy M and Vespignani A, 2004 *Phys. Rev.* E **70** 066149
[8] Newman M E J, 2004 *Phys. Rev.* E **70** 056131
[9] Newman M E J, 2001 *Proc. Nat. Acad. Sci.* **98** 404
[10] Newman M E J, 2001 *Phys. Rev.* E **64** 016131
[11] Albert R, Jeong H and Barabási A-L, 1999 *Nature* **401** 130
[12] Almaas E, Kovács B, Oltval Z N and Barabási A-L, 2004 *Nature* **427** 839
[13] Yook S H, Jeong H, Barabási A-L and Tu Y, 2001 *Phys. Rev. Lett.* **86** 5835
[14] Antal T and Krapivsky P L, 2005 *Phys. Rev.* E **71** 026103
[15] Bianconi G, 2005 *Europhys. Lett.* **71** 1029
[16] Wang W-X, Wang B-H, Hu B, Yan G and Ou Q, 2005 *Phys. Rev. Lett.* **94** 188702
[17] Goh K I, Kahng B and Kim D, 2005 *Phys. Rev.* E **72** 017103
[18] Wang W X, Hu B, Zhou T, Wang B H and Xie Y B, 2005 *Phys. Rev.* E **72** 046140
[19] Wang W X, Hu B, Wang B H and Yan G, 2005 *Phys. Rev.* E **73** 016133
[20] Zhang Z Z, Fang L J, Zhou S G and Guan J H, 2009 *Physica* A **388** 225
[21] Dorogvtsev S N and Mendes J F F, 2005 *AIP Conf. Proc.* **776** 29
[22] Zhang Z Z, Zhou S G, Fang L J, Guan J H and Zhang Y C, 2007 *Europhys. Lett.* **79** 38007
[23] Zhang Z Z, Zhou S G, Chen L C, Guan J H, Fang L J and Zhang Y C, 2007 *Eur. Phys. J.* B **59** 99
[24] Wu Z X, Xu X J and Wang Y H, 2005 *Phys. Rev.* E **71** 066124
[25] Barrat A, Barthélemy M and Vespignani A, 2005 *J. Stat. Mech.* P05003
[26] Mukherjee G and Manna S S, 2006 *Phys. Rev.* E **74** 036111
[27] Barabási A-L and Albert R, 1999 *Science* **286** 509
[28] Newman M E J, 2003 *SIAM Rev.* **45** 167
[29] Costa L D F, Rodrigues F A, Travieso G and Boas P R V, 2007 *Adv. Phys.* **56** 167
[30] Watts D J and Strogatz S H, 1998 *Nature* **393** 440
[31] Pastor-Satorras R and Vespignani A, 2001 *Phys. Rev. Lett.* **86** 3200
[32] Pastor-Satorras R and Vespignani A, 2001 *Phys. Rev.* E **63** 066117
[33] Liljeros F, Edling C R, Amaral L A N, Stanley H E and Aberg Y, 2001 *Nature* **411** 907
[34] Albert R, Jeong H and Barabási A L, 2000 *Nature* **406** 378
[35] Moreira A A, Andrade J S, Herrmann H J and Indekeu J O, 2009 *Phys. Rev. Lett.* **102** 018701
[36] Newman M E J and Girvan M, 2004 *Phys. Rev.* E **69** 026113
[37] Girvan M and Newman M E J, 2002 *Proc. Nat. Acad. Sci.* **99** 7821
[38] Newman M E J, 2009 *Phys. Rev. Lett.* **103** 058701
[39] Erdös P and Rényi A, 1959 *Publ. Math.* **6** 290
[40] Newman M E J, Strogatz S H and Watts D J, 2001 *Phys. Rev.* E **64** 026118
[41] Molloy M and Reed B, 1995 *Random Struct. Algorithms* **6** 161
[42] Newman M E J, 2002 *Phys. Rev. Lett.* **89** 208701
[43] Newman M E J, 2003 *Phys. Rev.* E **67** 026126
[44] Fronczak A, Fronczak P and Holyst J A, 2003 *Phys. Rev.* E **68** 046126